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On the Ono invariants of imaginary quadratic number fields

Stéphane R. Louboutin

Institut de Mathématiques de Luminy, UMR 6206, 163, avenue de Luminy, Case 907, 13288 Marseille Cedex 9, France

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ABSTRACT

J. Cohen, J. Sonn, F. Sairaiji and K. Shimizu proved that there are only finitely many imaginary quadratic number fields K whose Ono invariants Ono_K are equal to their class numbers h_K. Assuming a Restricted Riemann Hypothesis, namely that the Dedekind zeta functions of imaginary quadratic number fields K have no Siegel zeros, we determine all these K's. There are 114 such K's. We also prove that we are missing at most one such K. M. Ishibashi proved that if Ono_K is large enough compared with h_K, then the ideal class groups of K is cyclic. We give a short proof and a precision of Ishibashi's result. We prove that there are only finitely many imaginary quadratic number fields K satisfying Ishibashi's sufficient condition. Assuming our Restricted Riemann Hypothesis, we prove that the absolute values d_K of their discriminants are less than $2.3 \cdot 10^9$. We determine all these K's with $d_K \leqslant 10^6$. There are 76 such K's. We prove that there is at most one such K with $d_K \geqslant 1.8 \cdot 10^{11}$.

1. Introduction

Let $K = \mathbf{Q}(\sqrt{-d})$ denote an imaginary quadratic number field, where $d \geqslant 1$ is a square-free integer. Let d_K be the absolute value of its discriminant. Hence, $d_K = d$ if $d \equiv 3 \pmod 4$, and $d_K = 4d$ if $d \equiv 1, 2 \pmod 4$. Set $\omega_K := (1+\sqrt{-d})/2$ if $d \equiv 3 \pmod 4$, and $\omega_K := \sqrt{-d}$ otherwise. Set

$$E_K(n) := N_{K/\mathbf{Q}}(n+\omega_K) = \begin{cases} n^2+n+(d+1)/4 & \text{if } d \equiv 3 \pmod 4, \\ n^2+d & \text{otherwise.} \end{cases}$$

Set $Ono_K := 1$ if $d_K = 3, 4$, and $Ono_K := \max\{\Omega(E_K(n)),\ 0 \leqslant n \leqslant d_K/4 - 1\} \geqslant 1$ otherwise, where $\Omega(n) := m$ if $n > 1$ is a product of $m \geqslant 1$ (not necessarily distinct) primes. Let Cl_K and h_K be the

E-mail address: loubouti@iml.univ-mrs.fr.

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ideal class group and class number of K. Recall that h_K is odd if and only if $d_K = 4$, $d_K = 8$ or $d_K = p \equiv 3 \pmod 4$ is prime. We recall known results on the Ono invariant:

Theorem 1. *Let K denote an imaginary quadratic number field.*

1. *(See [Sas, Theorem 1].) It holds that $h_K \geqslant Ono_K$.*
2. *(E.g., see [Sas, Theorem], [L91, Théorème 2(a)].) $h_K = 1 \Leftrightarrow Ono_K = 1$.*
3. *(See [Sas, Theorem 2], [L91, Théorème 2].) $h_K = 2 \Leftrightarrow Ono_K = 2$.*
4. *(See [CS, Corollary 8].) If $h_K = 3$, then d_K is prime, $d_K \equiv 3 \pmod 4$ and $Ono_K = 3$.*
5. *(See [CS, Theorem 21].) There are only finitely many K's if d_K is prime, $d_K \equiv 3 \pmod 4$ and $Ono_K = 3$.*
6. *(See [GGL].) Assume the Generalized Riemann Hypothesis. Then, $h_K = 3 \Leftrightarrow d_K = p \equiv 3 \pmod 4$ is prime and $Ono_K = 3$.*
7. *(See [CS, Theorem 29], [GGL, Section 5], [Mo, Satz 5], [SS, Theorem 3.3].) Assume the Generalized Riemann Hypothesis. Then, $Ono_K \gg \frac{\log d_K}{\log\log d_K}$, and there are only finitely many K's of a given Ono invariant.*
8. *(Ishibashi's sufficient condition for Cl_K to be cyclic. See [Ish, Theorem, p. 613].) Assume that h_K is not square-free (otherwise, Cl_K is cyclic). Let $q_K \geqslant 2$ be the least prime whose square divides h_K. If*

$$Ono_K \geqslant q_K + h_K/q_K, \tag{1}$$

   then Cl_K is cyclic. Hence, if $h_K = Ono_K$, then Cl_K is cyclic. Moreover, assume that the 2-Sylow subgroup of Cl_K is nontrivial and cyclic (otherwise, Cl_K is not cyclic), which amounts to asking that d_K is divisible by at exactly two distinct primes. Then the same conclusion holds true for the least odd prime $q_K \geqslant 3$ whose square divides h_K.
9. *We have an explicit upper bound $Ono_K \ll \log d_K$ (see [SS, Theorem 3.4]) and the non-explicit asymptotic $\log h_K \sim \frac{1}{2}\log d_K$. Hence, there are only finitely many K's with $h_K = Ono_K$ (see [CS, Remark 7], [SS, Theorem 3.5]).*

Remarks 2. (E.g., see [Wat].)

1. $h_K = 1 \Leftrightarrow d_K \in \{3, 4, 7, 8, 11, 19, 43, 67, 163\}$.
2. $h_K = 2 \Leftrightarrow d_K \in \{15, 20, 24, 35, 40, 51, 52, 88, 91, 115, 123, 148, 187, 232, 235, 267, 403, 427\}$.
3. $h_K = 3 \Longleftrightarrow d_K \in \{23, 31, 59, 83, 107, 139, 211, 283, 307, 331, 379, 499, 547, 643, 883, 907\}$.

Here, we give a simple proof of Point 8 of Theorem 1 and we prove:

Theorem 3. *Let K denote an imaginary quadratic number field.*

1. *There are only finitely many K's with $h_K = Ono_K$ and all the K's with $h_K = Ono_K$ in the range $d_K \leqslant 2 \cdot 10^8$ are given in Table 1 (there are 114 such K's and they all satisfy $d_K \leqslant 4447$).*
2. *There is at most one K with $h_K = Ono_K$ and $d_K > 2 \cdot 10^8$.*
3. *Assume the Restricted Riemann Hypothesis: $\zeta_K(1 - (2/\log d_K)) \leqslant 0$ for the Dedekind zeta functions $\zeta_K(s)$ of all imaginary quadratic number fields K.[1] Then, there are no K's with $h_K = Ono_K$ and $d_K > 2 \cdot 10^8$.*

Theorem 4. *Let K denote an imaginary quadratic number field.*

1. *All K's with $d_K \leqslant 10^6$ for which Point 8 of Theorem 1 holds true are given in Table 2 (there are 76 such K's, and for $d_K = 8407$ and $d_K = 8767$, K does not satisfy Ishibashi's original criterium (1) with $q_K = 2$).*
2. *There are only finitely many such K's,*

[1] The Generalized Riemann Hypothesis is not known to hold true for any single number field. However, it holds that $\zeta_K(s) < 0$ for $0 < s < 1$ for a positive proportion of the imaginary quadratic number fields (see [CoSo]). Moreover, our Restricted Riemann Hypothesis is satisfied for all but very few imaginary quadratic number fields (see [HB, Theorem 3]).

3. *Under the assumption of our Restricted Riemann Hypothesis they satisfy* $d_K \leqslant 2.3 \cdot 10^9$.
4. *There is at most one such* K *with* $d_K \geqslant 1.8 \cdot 10^{11}$.

2. A simple proof of Ishibashi's result (Point 8 of Theorem 1)

Let $s_K \geqslant 1$ be the least number $s \geqslant 1$ such that an arbitrary sequence of length s of elements of Cl_K has a subsequence whose product is the principal class.

Lemma 5. *(See [Sas, Proof of Theorem 1].)* $s_K \geqslant Ono_K$.

Proof. We may assume that $d_K > 4$. By the definition of Ono_K, there exists an integer n such that $0 \leqslant n \leqslant d_K/4 - 1$ and $\Omega(E_K(n)) = Ono_K$. Then the primitive principal ideal $(n + \omega_K)$ is a product of Ono_K prime ideals $\mathcal{P}_i$, $1 \leqslant i \leqslant Ono_K$. We note that the ideal (ω_K) has the least norm among nontrivial primitive principal ideals. Since $N(n + \omega_K) < N(\omega_K)^2$, the ideal sequence $\mathcal{P}_i$, $1 \leqslant i \leqslant Ono_K$, does not have any proper subsequence whose product is the principal ideal. Thus $s_K > Ono_K - 1$. $\square$

We are now in a position to give a short proof of Ishibashi's result (Point 8 of Theorem 1). Let C_d denote a cyclic group of order $d \geqslant 1$. Suppose that the abelian group Cl_K is not cyclic. Then it is isomorphic to $C_{d_1} \times \cdots \times C_{d_r}$, with $r > 1$ and $1 < d_1 | \cdots | d_r$. By applying [Ols, Theorem 1] as $H = C_{d_1}$ and $K = C_{d_2} \times \cdots \times C_{d_r}$, we have $s_K \leqslant d_1 + h_K/d_1 - 1$. Since $q_K \leqslant d_1 \leqslant \sqrt{d_K}$, we have $d_1 + h_K/d_1 \leqslant q_K + h_K/q_K$. Together with Lemma 5 we have $Ono_K < q_K + h_K/q_K$. This completes the proof of the former part of Point 8. Moreover if the 2-Sylow subgroup of Cl_K is nontrivial and cyclic, all the d_i's but the last one d_r are odd. Thus the least odd prime whose square divides h_K is less than or equal to d_1. Similarly as above discussion, we prove the latter part of Point 8.

3. Proof of Theorem 3

Let us prove Point 1. We may assume that $d_K \neq 3$, 4. Set $m = Ono_K$. There exists n with $0 \leqslant n \leqslant d_K/4 - 1$ such that $\Omega(E_K(n)) = m$. It implies $2^m \leqslant E_K(n) \leqslant d_K^2/16$ and

$$Ono_K = m \leqslant 2\frac{\log(d_K/4)}{\log 2} \ll \log d_K,$$

as recalled in Point 9 of Theorem 1. The asymptotic on $\log h_K$ in Point 9 of Theorem 1 is nothing but Siegel's theorem (see [Sie]). As for the numerical check, recall that $h_K = 1$ for $d_K = 3$ and $d_K = 4$, and that otherwise we have

$$h_K = -\frac{1}{d_K} \sum_{\substack{x=1 \\ \gcd(x,d_K)=1}}^{d_K-1} x\chi_K(x),$$

where χ_K is the odd, primitive, quadratic Dirichlet character modulo d_K associated with K. A quick search on a microcomputer reveals that in the range $d_K \leqslant 2 \cdot 10^5$ we have $h_K = Ono_K$ if and only if d_K is one of the 114 values listed in Table 1. To deal with the range $2 \cdot 10^5 \leqslant d_K \leqslant 2 \cdot 10^8$, we use the much more efficient method for computing h_K developed in [L02] and test whether

$$h_K > 2\frac{\log(d_K/4)}{\log 2} \tag{2}$$

holds or not. In that case, we have $Ono_K < h_K$ and we do not compute Ono_K. There are no imaginary quadratic number fields K in the range $2 \cdot 10^5 \leqslant d_K \leqslant 2 \cdot 10^8$ for which (2) is not satisfied.

Now, let us prove Point 2. By Tatuzawa's result (see [Tat] and [L07]), if $h_K = Ono_K$, then for any $\epsilon \in (0, 1/2)$ and for $d_K \geqslant \max(e^{1/\epsilon}, e^{11.2})$, it holds that

$$\frac{0.655}{\pi}\epsilon d_K^{1/2-\epsilon} \leqslant h_K = Ono_K \leqslant 2\frac{\log(d_K/4)}{\log 2},$$

with at most one exception. Choosing $\epsilon = 0.053$, we obtain that there is at most one K with $h_K = Ono_K$ and $d_K > 1.6 \cdot 10^8$. Notice that the best known form of Tatuzawa's result (see [JL]) does not yield a better result.

Finally, let us prove Point 3. By [L90, Théorème 1], if $\zeta_K(1 - (2/\log d_K)) \leqslant 0$ and if $h_K = Ono_K$, then

$$\frac{\pi}{3e}\frac{\sqrt{d_K}}{\log d_K} \leqslant h_K = Ono_K \leqslant 2\frac{\log(d_K/4)}{\log 2},$$

which implies $d_K \leqslant 2.1 \cdot 10^6$.

Remarks 6. Assume that $d_K > 4$. If 2 is inert in K, then 2 does not divide any $E_K(n)$. If 2 is ramified in K, then 4 does not divide any $E_K(n)$. Hence, we have

$$Ono_K \leqslant \begin{cases} 2\log(d_K/4)/\log 3 & \text{if } d_K \equiv 3 \pmod 8, \\ 2\log(\sqrt{3/2}d_K/4)/\log 3 & \text{if } d_K \equiv 0 \pmod 4, \\ 2\log(d_K/4)/\log 2 & \text{if } d_K \equiv 7 \pmod 8. \end{cases} \tag{3}$$

It follows that there is at most one K with $h_K = Ono_K$, $d_K \not\equiv 7 \pmod 8$ and $d_K > 5 \cdot 10^7$ (choose $\epsilon = 0.056$).

Remarks 7. A quick search on a microcomputer reveals that in the range $d_K \leqslant 2 \cdot 10^5$, we have the following.

1. $Ono_K = 3$ and $h_K = 4$ for $d_K \in \{84, 120, 132, 168, 184, 195, 228, 280, 292, 312, 328, 340, 372, 408, 520, 532, 555, 595, 627, 708, 715, 723, 760, 763, 772, 795, 1003, 1012, 1243, 1387, 1435, 1555, \}$.
2. $Ono_K = 3$ and $h_K = 6$ for $d_K = 3763$.
3. $h_K = 3 \Leftrightarrow d_K = p \equiv 3 \pmod 4$ is prime and $Ono_K = 3 \Leftrightarrow d_K \in \{23, 31, 59, 83, 107, 139, 211, 283, 307, 331, 379, 499, 547, 643, 883, 907\}$.

4. Proof of Theorem 4

The proof of Theorem 4 is similar to that of Theorem 3, once we notice that condition (1) implies

$$Ono_K \geqslant 2\sqrt{h_K}.$$

Under the assumption of our Restricted Riemann Hypothesis, if $Ono_K \geqslant 2\sqrt{h_K}$ and $d_K > 4$, then

$$\frac{4\pi}{3e}\frac{\sqrt{d_K}}{\log d_K} \leqslant 4h_K \leqslant Ono_K^2 \leqslant 4\frac{\log^2(d_K/4)}{\log^2 2},$$

which implies $d_K \leqslant 2.3 \cdot 10^9$. Tatuzawa's result with $\epsilon = 0.038$ yields the at most one exception with $d_K \geqslant 1.8 \cdot 10^{11}$.

Remarks 8. These bounds are too large (even under the assumption of our Restricted Riemann Hypothesis) to enable us to determine all these K's satisfying (1). Using (3), we get that there is at most one K satisfying (1), $d_K \not\equiv 7 \pmod 8$ and $d_K > 1.6 \cdot 10^{10}$ (choose $\epsilon = 0.042$).

5. Tables

Table 1

d_K	h_K	d_K	h_K	d_K	h_K	d_K	h_K	d_K	h_K	d_K	h_K
3	1	56	4	151	7	283	3	568	4	1051	5
4	1	59	3	155	4	291	4	583	8	1087	9
7	1	67	1	163	1	295	8	619	5	1227	4
8	1	68	4	179	5	307	3	643	3	1255	12
11	1	71	7	183	8	323	4	667	4	1303	11
15	2	79	5	187	2	331	3	683	5	1411	4
19	1	83	3	199	9	347	5	691	5	1423	9
20	2	87	6	203	4	367	9	707	6	1507	4
23	3	88	2	211	3	379	3	727	13	1523	7
24	2	91	2	212	6	388	4	739	5	1527	14
31	3	103	5	219	4	403	2	751	15	1787	7
35	2	107	3	223	7	427	2	823	9	1867	5
39	4	115	2	227	5	443	5	827	7	2003	9
40	2	123	2	232	2	463	7	883	3	2143	13
43	1	127	5	235	2	467	7	907	3	2203	5
47	5	131	5	247	6	487	7	947	5	2251	7
51	2	136	4	259	4	499	3	955	4	3063	16
52	2	139	3	267	2	515	6	967	11	3343	19
55	4	148	2	271	11	547	3	1027	4	4447	17

Table 2

d_K	h_K	Ono_K	d_K	h_K	Ono_K	d_K	h_K	Ono_K
199	9	9	2228	18	10	9563	18	9
367	9	9	2887	25	12	10523	18	9
419	9	7	3035	18	10	10783	50	18
479	25	10	3107	18	9	11383	27	15
491	9	6	3271	27	13	12079	50	16
519	18	10	3543	18	14	13903	49	14
527	18	9	3943	27	13	16063	49	16
563	9	6	4399	50	17	17791	49	17
599	25	11	4568	18	9	19615	50	15
679	18	9	4591	49	16	19735	50	15
823	9	9	4903	27	12	20287	49	15
1087	9	9	5023	25	16	21107	25	11
1135	18	11	5492	18	9	21323	25	10
1187	9	6	5503	25	12	22327	50	16
1207	18	10	5843	25	11	23983	50	16
1231	27	12	6007	27	12	24511	50	19
1367	25	13	6415	50	15	26167	50	16
1383	18	10	7187	25	10	26383	50	15
1399	27	13	7283	25	11	26503	50	17
1423	9	9	7571	18	9	27127	49	16
1448	18	9	7715	18	9	30539	49	15
1687	18	9	7879	49	15	32167	50	15
1927	18	10	8407	36	15	40087	75	21
2003	9	9	8507	18	9	87823	98	21
2047	18	10	8727	50	17			
2167	18	12	8767	36	16			

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